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STABILITY OF UNSTEADY MOTION OF A VISCOUS FLUID BAND

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A brief derivation is presented in this paper for the small perturbation equations of arbitrary unsteady motion of a viscous incompressible fluid subjected to the action of surface forces. The stability of a viscous fluid band is studied on the basis of the equations obtained.

1. PERTURBATION EQUATIONS

We assume that the functions $u(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ are the velocity vector and pressure of a certain unsteady motion of a viscous incompressible fluid. The motion is defined in a domain $\Omega_t \subset R^3$ with boundary Γ_t. Within Ω_t, the $\mathbf{u}$, p satisfy the Navier–Stokes equations

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + (1/\rho)\nabla p = \nu\Delta\mathbf{u} + \mathbf{g}(\mathbf{x}, t); \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = 0,\ \mathbf{x} \in \Omega_t,\ t \geqslant 0, \tag{1.2}$$

and on Γ_t the conditions

$$(p_0 - p)\mathbf{n} + 2\rho\nu D(\mathbf{u})\mathbf{n} = 2\sigma H\mathbf{n}; \tag{1.3}$$

$$f_t + \mathbf{u}\cdot\nabla f = 0,\ \mathbf{x} \in \Gamma_t,\ t \geqslant 0, \tag{1.4}$$

where $\nu > 0$, ρ are the constant viscosity and density, n is the unit vector of the external normal to Γ_t; $\sigma > 0$ is the surface tension coefficient, D is the strain-rate tensor with elements $D_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ (i, j = 1, 2, 3); H is the mean curvature of the surface Γ_t (it is considered that $H > 0$ if Γ_t is convex within the fluid; p_0, $\mathbf{g}$ are the given external pressure and the mass force vector. Condition (1.3) expresses the equality of all forces acting on the free boundary while (1.4) denotes that Γ_t consists of the same particles (the equation $f(\mathbf{x}, t) = 0$ gives the free boundary Γ_t).

At the initial instant

$$\Omega_t|_{t=0} = \Omega,\ \mathbf{u}|_{t=0} = \mathbf{u}_0(\mathbf{x}),\ \Gamma_t|_{t=0} = \Gamma \tag{1.5}$$

and the consistency conditions are satisfied

$$\operatorname{div}\mathbf{u}_0 = 0,\ \boldsymbol{\tau}\cdot D(\mathbf{u}_0)\mathbf{n} = 0, \tag{1.6}$$

where τ is an arbitrary vector tangent to Γ.

Let us note that for $\sigma = 0$ the question of single-valued solvability of the problem posed is resolved affirmatively in [1], where $\mathbf{u}$, p, and Γ_t belong to certain Holder classes (see [2] also).

Let the solution $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ of the Navier–Stokes equations satisfying (1.3) and (1.4) on Γ_t, the initial conditions (1.5) and the consistency conditions (1.6) be known in the domain Ω_t. If $(\alpha_1, \alpha_2) \to \mathbf{x}(\alpha_1, \alpha_1, t)|_{t=0}$ is the parametric assignment of the initial surface $\Gamma \in C^3$ while the velocity vector $\mathbf{u}$ is a sufficiently smooth function, then [3] it can be considered that even Γ_t is parametrized by the same parameters (α_1, α_2): $\mathbf{x} = \mathbf{x}(\alpha_1, \alpha_2, t)$.

Let us consider another solution $\tilde{\mathbf{u}}$, $\tilde{p}$ in the domain $\tilde{\Omega}_t$ with the initial field $\tilde{\mathbf{u}}_0 = \mathbf{u}_0 + \mathbf{U}_0$, $\operatorname{div}\mathbf{U}_0 = 0$. Let $\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{X}(\mathbf{x}, t)$, $\mathbf{X}$ is the fluid particle displacement vector, $\mathbf{X}|_{t=0} = 0$, such that $\tilde{\Omega}_t|_{t=0} = \Omega$. We set

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$$\tilde{\mathbf{u}} = \mathbf{u}(\tilde{\mathbf{x}}, t) + \mathbf{U}(\tilde{\mathbf{x}}, t),\ \tilde{p} = p(\tilde{\mathbf{x}}, t) + P_1(\tilde{\mathbf{x}}, t),\ \tilde{f} = f(\tilde{\mathbf{x}}, t) + F(\tilde{\mathbf{x}}, t),$$

U, P_1, F are perturbations of the fundamental solution. Using the methods of [3], we can show that in a linear approximation

$$\frac{d\mathbf{U}}{dt} + \frac{\partial(\mathbf{u})}{v(\mathbf{x})}\mathbf{U} + \frac{1}{\rho}\nabla P_1 = \nu\Delta\mathbf{U}; \tag{1.7}$$

$$\operatorname{div}\mathbf{X} = 0; \tag{1.8}$$

$$\frac{d\mathbf{X}}{dt} = \frac{\partial(\mathbf{u})}{\partial(\mathbf{x})}\mathbf{X} + \mathbf{U}, \quad \mathbf{x} \in \Omega_t; \tag{1.9}$$

$$\frac{d}{dt}(F + |\nabla f| R) = 0, \quad R = \mathbf{n}\cdot\mathbf{X}, \quad \mathbf{x} \in \Gamma_t, \tag{1.10}$$

where $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$. By virtue of div **u** = 0, and taking account of (1.9) and the easily verifiable identity

$$\operatorname{div}\left[\frac{\partial(\mathbf{X})}{\partial(\mathbf{x})}\mathbf{u} - \frac{\partial(\mathbf{u})}{\partial(\mathbf{x})}\mathbf{X}\right] = \mathbf{u}\cdot\nabla(\operatorname{div}\mathbf{X}) - \mathbf{X}\cdot\nabla(\operatorname{div}\mathbf{u})$$

Eq. (1.8) is equivalent to the equation

$$\operatorname{div}\mathbf{U} = 0. \tag{1.8'}$$

In contrast to an ideal fluid, condition (1.3) has a vector form. Under our assumptions relative to the surface Γ_t, the triple of vectors **n**, $\mathbf{x}_{\alpha 1}$, $\mathbf{x}_{\alpha 2}$ forms a local basis which is generally not orthogonal, where the vectors $\mathbf{x}_{\alpha 1}$, $\mathbf{x}_{\alpha 2}$ are on a plane tangent to Γ_t. Consequently, (1.3) is equivalent to three scalar equations

$$p_0 - p + 2\rho\nu D(\mathbf{u})\mathbf{n}\cdot\mathbf{n} = 2\sigma H; \tag{1.11}$$

$$D(\mathbf{u})\,\mathbf{n}\cdot\mathbf{x}_{\alpha_i} = 0, \quad i = 1,2. \tag{1.12}$$

Let us set $\mathbf{X}|_{\Gamma_t} = R\mathbf{n} + X_1$, where X_1 is in a plane tangent to Γ_t. Writing the equalities (1.11) and (1.12) in the perturbed solution and linearizing, we obtain on Γ_t

$$-P_1 + 2\rho\nu D(\mathbf{U})\mathbf{n}\cdot\mathbf{n} = \left[\frac{\partial p}{\partial n} - 2\rho\nu\frac{\partial D(\mathbf{u})}{\partial n}\mathbf{n}\cdot\mathbf{n} + \sigma\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right)\right]R + \sigma\Delta_{\Gamma_t}R; \tag{1.13}$$

$$D(\mathbf{U})\,\mathbf{n}\cdot\mathbf{x}_{\alpha_i} + \frac{\partial D(\mathbf{u})}{\partial n}\mathbf{n}\cdot\mathbf{x}_{\alpha_i}R + D(\mathbf{u})\,\mathbf{n}\cdot\mathbf{n}\,R_{\alpha_i} + D(\mathbf{u})\,\mathbf{S}\cdot\mathbf{x}_{\alpha_i} = 0, \quad i = 1,2. \tag{1.14}$$

Here D(u) and D(U) are strain rate tensors of the fundamental and perturbed flows, R_1, R_2 are the principal radii of curvature of normal sections of the unperturbed surface Γ_t; Δ_{Γ_t} is the Laplace–Beltrami operator of the surface Γ_t. The vector S is defined by means of the equality

$$\mathbf{S} = \frac{1}{EG - F^2}\left[(FR_{\alpha_2} - GR_{\alpha_1})\mathbf{x}_{\alpha_1} + (FR_{\alpha_1} - ER_{\alpha_2})\mathbf{x}_{\alpha_2}\right], \tag{1.15}$$

E, G, and F are coefficients of the first quadratic form of Γ_t, $\mathbf{S}\cdot\mathbf{n} = 0$. The derivative with respect to the normal $\partial D(\mathbf{u})/\partial n$ is a matrix with the components $[\partial(\partial u_i/\partial x_j + \partial u_j/\partial x_i)/\partial n]/2$ (i, j = 1, 2, 3).

The relationships (1.13) and (1.14) are derived by long calculations using differential geometry formulas. We omit these calculations.

The initial conditions

$$\mathbf{U}|_{t=0} = \mathbf{U}_0,\ \operatorname{div}\mathbf{U}_0 = 0,\ F|_{t=0} = F_0,\ \mathbf{X}|_{t=0} = 0 \tag{1.16}$$

must be given for a complete definition of the perturbed motion.

Thus, the evolution of the perturbations is described by (1.7)-(1.9) with the boundary conditions (1.10), (1.13), (1.14) and the initial data (1.16).

The behavior of the perturbed solution for $t \to \infty$ or of other singular points is of interest in stability problems. Ordinarily the behavior of the boundary perturbations along the normal described by the function $R(\mathbf{x}, t) = \mathbf{n} \cdot \mathbf{X}$, $\mathbf{x} \in \Gamma_t$ is ordinarily of interest. It should be noted that the classical approach to the stability problems of stationary flows, for which solutions of the form $(U(\mathbf{x}), P(\mathbf{x})) \exp(\lambda t)$ are sought, is not appropriate for nonstationary flows since the perturbation equations obtained are not invariant with respect to a shift in time. In such cases the ordinary tendency is to separate out the space variables. The stability of one specific viscous fluid flow is investigated below.

2. TRANSFORMATION OF THE PERTURBATION EQUATIONS IN THE CASE OF A STRIP

The following is one of the exact solutions of the Navier–Stokes equations (1.1) and (1.2).

$$\mathbf{u} = \frac{k}{1+kt}(x, -y); \quad k = \text{const},$$
$$p = -\rho k^2(1+kt)^{-2}y^2 + \rho k^2 l^2 (1+kt)^{-4} - 2\rho\nu k(1+kt),\ l = \text{const} > 0. \tag{2.1}$$

It is easy to confirm that it satisfies the conditions (1.3), (1.4) and (1.6) if the lines $y = \pm l(1 + kt)^{-1}$ are taken on $\Gamma_t^{\pm}$. For $k > 0$ and $t \to \infty$, the strip transforms into the line $y = 0$. Since the free boundaries are straight lines, surface tension does not enter into the solution.

The motion defined by (2.1) is planar. We also limit ourselves to plane perturbations in the stability problem.

We have $D(\mathbf{u}) = k(1 + kt)^{-1}\,\mathrm{diag}\,(1, -1)$, $S = -R_x(1, 0)$, $\Delta_{\Gamma_t} R = R_{xx}$, $\mathbf{n} = (0, \pm 1)$. We introduce new independent dimensionless variables and functions

$$\tau = 1 + kt,\quad \xi = x/\tau l,\quad \eta = y\tau/l,\quad U = \tau U_1/kl,\quad V = U_2/kl\tau,\quad P = P_1/\rho k^2 l^2$$

(the variables ξ, η are Lagrange coordinates). Taking account of the formulas and the replacements obtained, the linearized problem (1.7)-(1.9), (1.10), (1.13), (1.14) and (1.16) is converted to the following for the fundamental motion (2.1):

$$U_\tau + P_\xi = \frac{1}{\mathrm{Re}}(\tau^{-2}U_{\xi\xi} + \tau^2 U_{\eta\eta}) \equiv \frac{1}{\mathrm{Re}} LU; \tag{2.2}$$

$$V_\tau + P_\eta = \frac{1}{\mathrm{Re}} LV; \tag{2.3}$$

$$U_\xi + \tau^4 V_\eta = 0, \quad |\eta| < 1, \quad \tau \geqslant 1; \tag{2.4}$$

$$-P + \frac{2}{\mathrm{Re}}\tau^2 V_\eta = -\frac{2}{\tau^3}R + \frac{\mathrm{We}}{\tau^2}R_{\xi\xi}; \tag{2.5}$$

$$\pm(U_\eta + V_\xi) - \frac{4}{\tau^2}R_\xi = 0, \quad \eta = \pm 1, \quad \tau \geqslant 1; \tag{2.6}$$

$$U = U_{1,0}(\xi, \eta),\ V = U_{2,0}(\xi, \eta),\ \partial U_{1,0}/\partial\xi + \partial U_{2,0}/\partial\eta = 0,\ \tau = 1. \tag{2.7}$$

Here $\mathrm{Re} = kl^2/\nu$ is the Reynolds number, and $\mathrm{We} = \sigma(\rho k^2 l^3)^{-1}$ is the Weber number. The normal component of the perturbation vector of the strip boundaries is determined by the equality

$$R = \pm\frac{1}{\tau}\int_1^\tau \tau^2 V\,d\tau, \quad \eta = \pm 1, \quad \tau \geqslant 1. \tag{2.8}$$

The problem (2.2)-(2.8) can be reduced to determining just the one function $V(\xi, \eta, \tau)$ as follows. Differentiating with respect to ξ, η in (2.2) and (2.3) and using the continuity equation (2.4), we find

$$\frac{\partial}{\partial\tau}(V_{\xi\xi} + \tau^4 V_{\eta\eta}) = \frac{1}{\mathrm{Re}} L(V_{\xi\xi} + \tau^4 V_{\eta\eta}), \quad |\eta| < 1. \tag{2.9}$$

Taking of (2.8) we rewrite the boundary conditions (2.6) in the form

$$\pm(V_{\xi\xi} - \tau^4 V_{\eta\eta}) \mp \frac{4}{\tau^3}\int_1^\tau \tau^2 V_{\xi\xi}\,d\tau = 0, \quad \eta = \pm 1. \tag{2.10}$$

Furthermore, differentiating (2.5) twice with respect to ξ and replacing $P_{\xi\xi}$ from (2.2) and U_ξ from (2.4), we obtain

$$\frac{1}{\mathrm{Re}} L(\tau^4 V_\eta) - \frac{\partial}{\partial\tau}(\tau^4 V_\eta) + \frac{2\tau^2}{\mathrm{Re}} V_{\eta\xi\xi} = \pm\left[-\frac{2}{\tau^4}\int_1^\tau \tau^2 V_{\xi\xi} d\tau + \frac{\mathrm{We}}{\tau^3}\int_1^\tau \tau^2 V_{\xi\xi\xi\xi} d\tau\right], \quad \eta = \pm 1. \tag{2.11}$$

If the function $V(\xi, \eta, \tau)$ is known as a solution of the problem of (2.9)-(2.11) with the initial data $U_{2,0}(\xi, \eta)$, then the behavior of the perturbations of the strip free boundaries is determined from (2.8).

3. CONSTRUCTION OF THE SOLUTION AND ITS ASYMPTOTIC ANALYSIS

The variables (η, τ), ξ separate in the problem (2.9)-(2.11). We shall consider the function V periodic in ξ with period h. For one harmonic we set $V = \Phi(\eta, \tau)\exp(in\xi)$, $n = n_1\pi l/h$, $n_1 = 1, 2, \ldots$ and we introduce the new function $\Psi(\eta, \tau)$ by the equality

$$\Psi = \tau^4\Phi_{\eta\eta} - n^2\Phi. \tag{3.1}$$

Here Ψ satisfies the one-dimensional heat conduction equation

$$\Psi_\tau = \frac{1}{\mathrm{Re}}\tau^2\Psi_{\eta\eta} - \frac{n^2}{\mathrm{Re}\,\tau^2}\Psi. \tag{3.2}$$

This results from (2.9). In terms of the functions $\Phi(\eta, \tau)$, $\Psi(\eta, \tau)$ the boundary conditions (2.10) and (2.11) take the form

$$\Psi + 2n^2\Phi = \frac{4n^2}{\tau^3}\int_1^\tau \tau^2\Phi d\tau, \quad \eta = \pm 1; \tag{3.3}$$

$$\frac{\tau^2}{\mathrm{Re}}\Psi_\eta - \frac{\partial}{\partial\tau}(\tau^4\Phi_\eta) - \frac{2\tau^2 n^2}{\mathrm{Re}}\Phi_\eta = \pm q(\tau)\int_1^\tau \tau^2\Phi d\tau, \quad \eta = \pm 1, \tag{3.4}$$

where

$$q(\tau) = 2n^2/\tau^4 + n^4\mathrm{We}/\tau^3. \tag{3.5}$$

Let

$$\Psi|_{\eta=1} = \Psi_1(\tau), \quad \Psi|_{\eta=-1} = \Psi_2(\tau), \quad \Psi|_{\tau=1} = \Psi_0(\eta) = U_{2,0\eta\eta} - n^2 U_{2,0}. \tag{3.6}$$

Then the first initial-boundary value problem (3.2), (3.6) uniquely determines the function $\Psi(\eta, \tau) = \tilde{\Psi}(\eta, \tau, \Psi_0, \Psi_1, \Psi_1)$. Then $\Phi(\eta, \tau) = \tilde{\Phi}(\eta, \tau, \Psi_0, \Psi_1, \Psi_2, C_1, C_2)$ is found from (3.1) with the unknown functions $C_1(\tau)$, $C_2(\tau)$. Substitution into the boundary conditions (3.3), (3.4) results in a system of four integrodifferential equations to determine the unknown functions. The function $\Psi_1(\tau)$, $\Psi_2(\tau)$, $C_1(\tau)$, $C_2(\tau)$. The function $\Phi(\eta, \tau)$ is thereby found, meaning also the amplitude of the normal component of the perturbation vector $R_{n_1}(\tau) = R\exp(-in\xi)$ from (2.8).

Let us note the following property of the formulated problem that results from the boundary conditions (3.3), (3.4) and the solution of the mixed problem (3.2), (3.6): it is possible to limit onself to seeking just even or odd solutions in the variable η. Since from (3.1)

$$\Phi(\eta, \tau) = C_1(\tau)\,\mathrm{sh}\frac{n}{\tau^2}\eta + C_2(\tau)\,\mathrm{ch}\frac{n}{\tau^2}\eta - \frac{1}{n\tau^2}\int_0^\eta \mathrm{sh}\frac{n}{\tau^2}(\mu - \eta)\Psi(\mu, \tau)d\mu, \tag{3.7}$$

for even perturbations it is necessary to set $C_1(\tau) = 0$, $\Psi_1(\tau) = \Psi_2(\tau)$, $U_{2,0}(\eta) = U_{2,0}(-\eta)$, and for odd $C_2(\tau) = 0$, $\Psi_1(\tau) = -\Psi_2(\tau)$, $U_{2,0}(\eta) = -U_{2,0}(-\eta)$. It is sufficient to take the boundary conditions (3.3), (3.4) for $\eta = 1$ here.

It can be shown that for odd perturbations the function $\Psi(\eta, \tau)$ has the form

$$\Psi(\eta, \tau) = \exp(n^2/\mathrm{Re}\,\tau)\sum_{m=1}^{\infty}\left[2\pi(-1)^m m\int_0^\gamma f_{2m}(\gamma - \mu)w(\mu)d\mu + c_m f_{2m}(\gamma)\right]\sin m\pi\eta, \tag{3.8}$$

and for even

$$\Psi = \exp(n^2/\mathrm{Re}\,\tau) \sum_{m=0}^{\infty} \left[(2m+1)\pi(-1)^m \int_0^{\gamma} f_{2m+1}(\gamma-\mu)\, w(\mu) d\mu \;+ d_m f_{2m+1}(\gamma) \right] \cos(2m+1)\pi\eta, \quad (3.9)$$

where

$$f_m(\gamma) = \exp(-m^2\pi^2\gamma/4),\; \gamma = (\tau^3-1)/3\mathrm{Re},$$
$$w(\gamma) = \exp[-n^2/\mathrm{Re}\,\tau(\gamma)]\Psi_1(\tau(\gamma)), \quad (3.10)$$

c_m, d_m are Fourier series coefficients of the odd or even initial function $\exp(-n^2/\mathrm{Re})\Psi_0(\eta)$

It is convenient to introduce a new unknown function

$$B(\tau) = \int_1^{\tau} \tau^2 \Phi|_{\eta=1}\, d\tau. \quad (3.11)$$

in place of the function $C_1(\tau)$ (or $C_2(\tau)$). Substitution of the expressions (3.8), (3.9), (3.11) into (3.3), (3.4) results after sufficiently awkward calculations in expressions with $B(\tau)$, $\Psi_1(\tau)$. For odd perturbations we have the system

$$q(\tau)B = \frac{n}{\mathrm{Re}} \operatorname{cth}\frac{n}{\tau^2}\Psi_1 - n\left(\operatorname{cth}\frac{n}{\tau^2}B'\right)' - \frac{2n^3}{\mathrm{Re}\,\tau^3}\operatorname{cth}\frac{n}{\tau^2}B' - 2\pi\int_0^{\gamma} K^-(\tau,\gamma-\mu)\, w(\mu)\, d\mu + Q^-(\tau); \quad (3.12)$$

$$\Psi_1 = -2n^2(B/\tau^2)', \quad (3.13)$$

where

$$K^-(\tau,\gamma-\mu) = \exp(n^2/\mathrm{Re}\,\tau)\sum_{m=1}^{\infty} mA_m(\tau) f_{2m}(\gamma-\mu),$$

$$Q^-(\tau) = \exp(n^2/\mathrm{Re}\,\tau)\sum_{m=1}^{\infty}(-1)^m A_m(\tau) f_{2m}(\gamma) c_m,$$

$$A_m(\tau) = \frac{2n^2 m\pi}{n^2+m^2\pi^2\tau^4}\left(\frac{\tau^2}{\mathrm{Re}} + \frac{2\tau^3}{n^2+m^2\pi^2\tau^4}\right).$$

For even perturbations we obtain

$$q(\tau)B = \frac{n}{\mathrm{Re}} \operatorname{th}\frac{n}{\tau^2}\Psi_1 - n\left(\operatorname{th}\frac{n}{\tau^2}B'\right)' - \frac{2n^3}{\mathrm{Re}\,\tau^2}\operatorname{th}\frac{n}{\tau^2}B' - \pi\int_0^{\gamma} K^+(\tau,\gamma-\mu)\, w(\mu)\, d\mu - Q^+(\tau); \quad (3.14)$$

$$\Psi_1 = -2n^2(B/\tau^2)', \quad (3.15)$$

where

$$K^+(\tau,\gamma-\mu) = \exp(n^2/\mathrm{Re}\,\tau)\sum_{m=0}^{\infty}(2m+1) D_m(\tau) f_{2m+1}(\gamma-\mu),$$

$$Q^+(\tau) = \exp(n^2/\mathrm{Re}\,\tau)\sum_{m=0}^{\infty}(-1)^m D_m(\tau) f_{2m+1}(\gamma) d_m,$$

$$D_m(\tau) = \frac{4n^2(2m+1)\pi}{4n^2+(2m+1)^2\pi^2\tau^4}\left[\frac{\tau^2}{\mathrm{Re}} + \frac{8\tau^3}{4n^2+(2m+1)^2\pi^2\tau^4}\right].$$

The functions $q(\tau)$, $f_m(\gamma)$, $w(\gamma)$ are here determined from (3.5) and (3.10) while $\gamma = (\tau^3 - 1)/3\mathrm{Re}$. According to (2.8) and (3.11), the amplitude of the normal perturbation vector component has the form $R_{n_1} = B(\tau)/\tau$, $n_1 = 1, 2, \ldots$

It can be shown that as $\tau \to \infty$ the system (3.12), (3.13) and (3.14), (3.15) reduces to a system of three ordinary first-order differential equations with irregular singularities. This system is inhomogeneous and the rank of its singularity $\tau = \infty$ is three [4]. We present here the principal terms of the asymptotic of the function $R_{n_1}(\tau)$. For odd perturbations for all $\sigma \geq 0$

$$R_{n_1} \sim \frac{a_1}{\tau} + \frac{a_2}{\tau^2} + \frac{(-1)^m n^2 \operatorname{Re} c_m}{m^5 \pi^5 \tau^6} \exp\left(-\frac{m^2\pi^2}{3\operatorname{Re}}\tau^3\right), \quad \tau \to \infty, \tag{3.16}$$

if $c_i = 0$, $i = 1, \ldots, m-1$. The behavior of the even perturbations depends on the surface tension σ. Namely, for $\sigma = 0$, $\tau \to \infty$

$$R_{n_1} \sim a_1\tau + \frac{a_2}{\tau^2} - \frac{64 \operatorname{Re} (-1)^m d_m}{3(2m+1)^5 \pi^5 \tau} \exp\left[-\frac{(2m+1)^2\pi^2}{12\operatorname{Re}}\tau^3\right], \tag{3.17}$$

while for $\sigma > 0$, $\tau \to \infty$

$$R_{n_1} \sim \tau^{1/4} \left\{a_1 \cos(2n\sqrt{\operatorname{We}\tau}) + a_2 \sin(2n\sqrt{\operatorname{We}\tau}) - \frac{64n^2 \operatorname{Re}(-1)^m d_m}{n\sqrt{\operatorname{We}}(2m+1)^5 \pi^5 \tau^{5/4}} \exp\left[-\frac{(2m+1)^2\pi^2}{12\operatorname{Re}}\tau^3\right]\right\}, \tag{3.18}$$

if $d_i = 0$, $i = 1, \ldots, m-1$; $m = 0, 1, \ldots$ The a_1, a_2 in (3.16)-(3.18) are constants.

Deductions about the flow stability of the strip (2.1) as $t \to \infty (\tau \to \infty)$ can be made on the basis of the asymptotics (3.16)-(3.18). The odd perturbations are stable, where the initial perturbations damp out exponentially. It is interesting to note that in the ideal fluid scheme analogous vortical initial data destabilize the free boundary [5]. This deduction does not contradict that expressed above since the asymptotic (3.16) is not uniform in ν as $\tau \to \infty$ (or equivalently, Re), when $\nu \to 0$ (Re $\to \infty$). Even perturbations always grow, as follows from (3.17) and (3.18), although the surface tension reduces the instability somewhat without eliminating it completely.

Thus, even velocity perturbations along the y axis are most dangerous. The so-called "hose-like instability" corresponds to them.

In conclusion we note that as $\tau \to 0$ ($t \to -1/k$, $k < 0$ stability $R_{n_1} \sim a\tau$, a = const.

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